AFCRL 353
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On the Theory of Nonstationary Light Scattering

I. A. KLIMISHIN

Ukr. Fiz. Zh., vol. 5, No. 5, 1960, pp. 620 - 627

341 200

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Post Office Box 35
West Newton 65, Massachusetts

K - 249

S/A 2, AF19(604) - 5969

May, 1961

Prepared
for

ELECTRONICS RESEARCH DIRECTORATE
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

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On the Theory of Nonstationary Light Scattering

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A strong shockwave-front which moves in the atmosphere of the earth (or even in a starry envelope or in interstellar space) radiates light; the radiant energy is here absorbed by the gas ahead of the wavefront to some degree. Radiant energy from a strong shockwave front considerably affects the magnitude of the temperature jump at the wavefront and determines its configuration. The effect of interference from radiation on the characteristics of strong "terrestrial" shockwaves was considered by IA. B. Zel'dovich [1] and by IU. P. Raizer [2]. The effect of radiation from a shockwave front which moves in the shell of a star, on the magnitude of the temperature jump at the wavefront is analyzed in [3]. The results of the above work confirm the necessity of taking into account the interaction of strong shockwaves with radiation independently of whether they move in the atmosphere of the earth, a star or in interstellar space.

But up to recently, it was customary to solve a system of equations of motion together with the equations of radiation transfer in investigations of these problems. On the whole, this system is complicated and is solved only by numerical methods. Undoubtedly the use of contemporary methods of the theory of light scattering can afford new possibilities in this direction. But only the case of light scattering in a medium whose boundaries remain fixed has been solved in problems of this theory. In the case of light scattering ahead of or behind a shockwave front, the boundary of the medium which is scattered moves relative to the gas. Hence, before applying the modern theory of light scattering to gasdynamics problems, it is necessary to develop a theory of nonstationary light scattering in a medium with moving boundaries.

PRINCIPAL STATEMENTS OF THE THEORY OF NONSTATIONARY LIGHT SCATTERING

Of all the methods of light scattering theory, the method of determining the probability that a quantum would leave a medium which was proposed by V. V. Sobolev [4] seemed most suitable to us. From this method, the probability $p(t,\tau)$ is found that a light quantum absorbed to the optical depth τ at zero time will leave within the time t. Two cases hence exist: 1) Most of the time the quantum is in the absorbed state; 2) most of the time the quantum resides on the path between two absorptions. The first case has meaning in the atmosphere of the earth and in the envelopes of stars, the second in interstellar space and in the upper layers of the atmosphere of stars.

V. V. Sobolev formulated integral and equivalent differential equations for the probability that a quantum would exit from the medium in both cases. In particular, the solution for the function $p(\tau,u)$, where $u = \frac{t}{t_1}$ is nondimensional time, is

$$p(\tau,u) = \frac{2}{\pi}\lambda\int_0^\infty (x\cos x\tau + \sin x\tau)e^{-\left(1-\frac{\lambda}{1+x^2}\right)u}\frac{x\,dx}{(1+x^2)^2}, \tag{1}$$

in the first case, where λ is the probability of the survival of the quantum after the act of absorption; t_1 the mean time spent by the quantum on the path between two scattering acts.

For practical purposes the determination of the function $p(\tau)$, the probability that a light quantum absorbed at the optical depth τ will generally leave the medium, and of the function $Z(t)$, which determines the mean time the quantum spends in the medium, is of considerable interest. These functions are determined from the expressions

$$P(\tau) = \int_0^\infty p(\tau,u)\,du; \tag{2}$$

$$Z(\tau) = \int_0^\infty u\,p(\tau,u)\,du. \tag{3}$$

V. V. Sobolev found integral and equivalent differential equations for $P(\tau)$ and $Z(\tau)$ as well as the solutions for these functions in the case of a bounded medium and for $\lambda = 1$. The function $P(\tau)$ for a semi-infinite medium is found

from

(4) $$P(\tau) = (1 - \sqrt{1 - \lambda})e^{-\sqrt{1-\lambda}\tau}.$$

Let us note that for the case of a semi-infinite medium we have found the solution of the appropriate differential equation for $Z(\tau)$ as

(5) $$\frac{d^2Z}{d\tau^2} = (1 - \lambda)Z - \lambda P$$

with the boundary conditions $Z_{(0)} = Z'(0) + \lambda$.

This solution can be written thus

(6) $$Z(\tau) = \frac{\lambda}{2\sqrt{1-\lambda}}[1 + (1 - \sqrt{1 - \lambda})\tau]e^{-\sqrt{1-\lambda}\tau}.$$

In the expressions presented, $\tau = knx$ is the optical depth, where k is the absorption coefficient per particle; n the number of particles per unit volume and x a geometric coordinate.

This well-behaved and completely conclusive theory can also be applied to the case of light scattering with a moving boundary. In this case, the optical depth will be considered as the position of the boundary at a given instant. Let us imagine the medium to be semi-infinite (in optical thickness), i.e., bounded on one side by a boundary which moves with the constant velocity $v = \frac{d\tau}{du}$ either in or out of the medium. Hence, let us imagine that the light is not scattered on the other side of the moving boundary. For simplicity of the computations, let us be limited to the case of a one-dimensional medium. Let us also imagine that the probability of survival of a quantum after scattering λ is independent of the optical thickness and that the probability of scattering on both sides is the same.

PROBABILITY THAT A QUANTUM WILL LEAVE A MEDIUM WITH A MOVING BOUNDARY AT THE SIDE OF THE MEDIUM

A boundary moving with velocity v will traverse a path uv within the time u. Hence, in this case it is required to search separately for the probabilities that a quantum will leave the optical depths $\tau < uv$ and $\tau > uv$. If $\tau > uv$, then the probability that the quantum will leave without scattering within the time u will equal $\frac{\lambda}{2}e^{-(\tau-uv)-u}$ and the probability that the quantum will leave after scattering can be found by multiplying the probability of the transition of

a quantum from the depth τ to the depth τ' within the time u', which is $\frac{\lambda}{2}e^{-|\tau-\tau'|-u'}$ and the probability that the quantum will leave the depth τ' at the time $u - u' - p(\tau'-u'v,u-u')$. It is hence required to take into consideration that actually the distance from the quantum to the moving boundary at time u' will be $\tau' - u'v$. Integration should be over all τ from $u'v$ to infinity. Hence, making the change of variable $\tau'' = \tau' - u'v$, we find that for all $\tau > uv$

$$p(\tau,u) = \frac{\lambda}{2} e^{-\tau+uv-u} + \frac{\lambda}{2}\int_0^u e^{-u'} du' \int_0^\infty e^{-|\tau-\tau''-u'v|} p(\tau'',u-u')d\tau''. \tag{7}$$

The expression for $p(\tau,u)$ for the optical depth $\tau < uv$ has a less complex form. The probability that a quantum will exit without scattering is $\frac{\lambda}{2}e^{-u}$. The probability that a quantum will exit after scattering is composed of two factors: the first determines the probability that a quantum will exit after scattering with transitions from the optical depth τ to the depth τ' during the time the movable boundary has not yet achieved the optical depth τ, i.e., in the time $0 \leq u \leq \frac{\tau}{v}$; the second factor takes into account the probability that a light quantum will exit from the medium which has remained in the absorbed state a time greater than $\frac{\tau}{v}$, has been carried out of the medium along with an atom, then reradiated to the side of the medium and has been absorbed at the optical depth τ'. Thus, the integral equation for $p(\tau,u)$ for all $\tau < uv$ is

$$p(\tau,u) = \frac{\lambda}{2} e^{-u} + \frac{\lambda}{2}\int_0^{\frac{\tau}{v}} e^{-u'} du' \int_0^\infty e^{-|\tau-\tau''-u'v|} p(\tau'',u-u')d\tau'' + \tag{8}$$
$$+ \frac{\lambda}{2}\int_{\frac{\tau}{v}}^{u} e^{-u'} du' \int_0^\infty e^{-\tau'} p(\tau',u-u')d\tau'.$$

The appropriate differential equation and boundary conditions can be obtained from (7) and (8). The differential equation can be solved by separation of variables. However, it should be noted that these equations are very awkward and the solutions obtained have a form such that it would be difficult to obtain a clear physical picture and estimate of the quantities for comparison with the results of observations. Hence, instead of the solved equations (7) and (8), it

is expedient to obtain appropriate equations from them to determine the probability $P(\tau)$ that a quantum would leave the medium at some instant and $Z(\tau)$ the average time the quantum stays in the medium after ist to the depth τ, which are defined by (2) and (3).

In the case we are considering

$$P(\tau) = \int_0^\infty p(\tau,u)du = \int_0^{\frac{\tau}{v}} p_1(\tau,u)du + \int_{\frac{\tau}{v}}^\infty p_2(\tau,u)du, \tag{9}$$

where $p_1(\tau,u)$ is the expression (7), which defines the probability considered at $u<\frac{\tau}{v}$ and $p_2(\tau,u)$ for $u>\frac{\tau}{v}$. Substituting (7) and (8) into (9), we obtain after integration with respect to u

$$\begin{aligned} P(\tau) = {} & \frac{\lambda}{2(1-v)}\left[e^{-\tau} - ve^{-\frac{\tau}{v}}\right] + \frac{\lambda}{2(1-v)}\int_0^\tau e^{-(\tau-\tau')}P(\tau')d\tau' + \\ & + \frac{\lambda}{2(1+v)}\int_0^\infty e^{-(\tau'-\tau)}P(\tau')d\tau' - \frac{\lambda v}{1-v^2}\int_0^\tau e^{-\frac{\tau-\tau'}{v}}P(\tau')d\tau' + \\ & + \frac{\lambda v}{2(1+v)}\int_0^\infty e^{-\tau'}P(\tau')d\tau'. \end{aligned} \tag{10}$$

From this equation, we have the differential equation determining $P(\tau)$:

$$\frac{d^3P}{d\tau^3} + \frac{1}{v}\cdot\frac{d^2P}{d\tau^2} - \frac{dP}{d\tau} - \frac{1-\lambda}{v}P = 0. \tag{11}$$

Hence, it follows that the function $P(\tau)$ can be written as

$$P(\tau) = \sum c_i e^{-k_i\tau}, \tag{12}$$

where the c_i are constants determined from the boundary conditions, k_i the roots of the characteristic equation

$$k^3 - \frac{1}{v}k^2 - k + \frac{1-\lambda}{v} = 0. \tag{13}$$

Of the three roots of this equation, one is negative, it is discarded since the solution must remain bounded as $\tau \to \infty$.

Substituting (12) into (10), the integration can be performed, the

coefficients of the various exponents equated to zero and conditions obtained to determine the constants c_i :

(14)
$$\sum_i \frac{c_i}{1-k_i} = 1; \quad \frac{1-v}{1+v} \sum_i \frac{c_i}{1+k_i} - \frac{2}{1+v} \sum_i \frac{c_i}{k_i - \frac{1}{v}} = 1 .$$

Formulas (12) - (14) completely determine the solution of our problem; thus

(15)
$$P(\tau) = \frac{k_3(1-k_1^2)(1-k_1 v)}{(k_3-k_1)(1+k_1k_3v)} e^{-k_1\tau} + \frac{k_1(1-k_3^2)(1-k_3v)}{(k_1-k_3)(1+k_1k_3v)} e^{-k_3\tau},$$

where the k_i found from the characteristic equation equal:

(16)
$$\left.\begin{aligned} k_1 &= \frac{1}{3v}\left[1+2\sqrt{1+3v^2}\cos\left\{\frac{1}{3}\operatorname{arc\,cos}\frac{1}{2}\frac{2+9v^2(3\lambda-2)}{(1+3v^2)^{\frac{3}{2}}}\right\}\right] \\ k_3 &= \frac{1}{3v}\left[1-2\sqrt{1+3v^2}\cos\left\{\frac{\pi}{3}+\frac{1}{3}\operatorname{arc\,cos}\frac{1}{2}\frac{2+9v^2(3\lambda-2)}{(1+3v^2)^{\frac{3}{2}}}\right\}\right] \end{aligned}\right\}.$$

An approximate formula for $P(\tau)$ can be obtained. Thus, for $v \ll 1$

(17)
$$k_1 \approx \frac{1}{v} - (1-\lambda)v; \quad k_3 = \sqrt{1-\lambda} + \frac{(1-\lambda)v}{2};$$

$$P(\tau) \approx \frac{(1-\lambda)^{\frac{3}{2}}}{1+\sqrt{1-\lambda}} v e^{-\frac{\tau}{v}} + (1-\sqrt{1-\lambda})e^{-\sqrt{1-\lambda}\tau} .$$

and for $v \gg 1$

(18)
$$k_1 \approx 1 + \frac{\lambda}{2v}; \quad k_3 \approx \frac{1-\lambda}{v};$$

$$P(\tau) \approx \frac{\lambda}{2-\lambda} e^{-\frac{1-\lambda}{v}\tau} .$$

Similarly, we find the integral equation for $Z(\tau)$

(19)
$$Z(\tau) = \int_0^\infty u p(\tau,u)du = \int_0^{\frac{\tau}{v}} p_1(\tau,u)u\,du + \int_{\frac{\tau}{v}}^\infty p_2(\tau,u)u\,du.$$

Substituting the approximate values $p_1(\tau,u)$ and $p_2(\tau,u)$ according to (7) and (8) and integrating with respect to u , we find

$$Z(\tau) = \frac{\lambda}{2(1-v)^2} e^{-\tau} + \frac{\lambda}{2} \cdot \frac{\tau(v-1)+v(v-2)}{(1-v)^2} e^{-\frac{\tau}{v}} +$$

$$+ \frac{\lambda}{2(1-v)^2} \int_0^\tau e^{-(\tau-\tau')} P(\tau')d\tau' + \frac{\lambda}{2(1+v)^2} \int_\tau^\infty e^{-(\tau'-\tau)} P(\tau')d\tau' -$$

(20)

$$- \frac{2\lambda v}{(1-v^2)^2} \int_0^\tau e^{-\frac{\tau-\tau'}{v}} P(\tau')d\tau' - \frac{\lambda}{1-v^2} \int_0^\tau (\tau-\tau')e^{-\frac{\tau-\tau'}{v}} P(\tau')d\tau' +$$

$$+ \frac{\lambda}{2}\left[\frac{v(v+2)}{(1+v)^2} + \frac{\tau}{1+v}\right] e^{-\frac{\tau}{v}} \int_0^\infty e^{-\tau'} P(\tau')d\tau' + \frac{\lambda}{2(1-v)} \int_0^\tau e^{-(\tau-\tau')} Z(\tau')d\tau' +$$

$$+ \frac{\lambda}{2(1+v)} \int_\tau^\infty e^{-(\tau'-\tau)} Z(\tau')d\tau' - \frac{\lambda v}{1-v^2} \int_0^\tau e^{-\frac{(\tau-\tau')}{v}} Z(\tau')d\tau' +$$

$$+ \frac{\lambda v}{2(1+v)} e^{-\frac{\tau}{v}} \int_0^\infty e^{-\tau'} Z(\tau')d\tau' .$$

It is easy to obtain the differential equation for $Z(\tau)$ from this equation

(21) $$\frac{d^3Z}{d\tau^3} + \frac{1}{v}\frac{d^2Z}{d\tau^2} - \frac{dZ}{d\tau} - \frac{1-\lambda}{v} Z = \frac{1}{v}\left[\frac{d^2P}{d\tau^2} - P\right],$$

from which it is seen that the solution for the function $Z(\tau)$ can be represented in the form

(22) $$Z(\tau) = \sum (A_i + B_i\tau)e^{-k_i\tau}.$$

Substituting (22) and (12) into (20) and integrating, we find

(23) $$\left.\begin{aligned} B_i &= \frac{k_i^2 - 1}{3k_i^2 v - 2k_i - v} C_i \\ A_i &= \frac{2v^3(k_i - 1)^2 - (1+v)(k_i v - 1)^2}{(k_i - 1)(k_i v - 1)[(1+v)(k_i v - 1) - 2v^2(k_i - 1)]} B_i \end{aligned}\right\};$$

The solution for $Z(\tau)$ is now written as

$$Z(\tau) = \sum_{i=1,3} \frac{k_m(k_i+1)(k_i^2-1)[2v^3(k_i-1)^2 - (1+v)(k^iv - 1)^2]}{(k_m-k_i)(1+k_ik_mv)(3k_i^2v-2k_i-v)[(1+v)(k_iv-1)-2v^2(k_i-1)]} \times \tag{24}$$
$$\times \left[1+ \frac{(k_i-1)(k_iv-1)[(1+v)(k_iv-1)-2v^2(k_i-1)]}{2v^3(k_i-1)^2 - (1+v)(k_iv-1)^2}\tau\right]e^{-k_i\tau},$$

where $m=3$ for $i=1$ and $m=1$ for $i=3$; k_1 and k_2 are determined by (16).

Expression (24) is considerably simplified for $v \gg 1$ and reduces to

$$Z(\tau) \approx \frac{1}{2-\lambda}\left[1+\frac{\lambda}{v}\tau\right]e^{-\frac{1-\lambda}{v}\tau}. \tag{25}$$

PROBABILITY THAT A QUANTUM WILL LEAVE THE MEDIUM FOR MOTION OF THE MEDIUM BOUNDARY

Starting from considerations analogous to those presented above, we find for the probability that a quantum will leave the medium if the boundary of the medium moves

$$p(\tau,u) = \frac{\lambda}{2}e^{-\tau-uv-u} + \frac{\lambda}{2}\int_0^u e^{-u'}du'\int_0^\infty e^{-|\tau-\tau''+u'v|}p(\tau'',u-u')d\tau'', \tag{26}$$

and integrating this expression with respect to u , we obtain an integral expression for the probability $P(\tau)$

$$P(\tau) = \frac{\lambda}{2(1+v)}e^{-\tau} + \frac{\lambda}{2(1+v)}\int^\tau e^{-(\tau-\tau')}P(\tau')d\tau' + \tag{27}$$
$$+ \frac{\lambda}{2(1-v)}\int_\tau^\infty e^{-(\tau'-\tau)}P(\tau')d\tau' - \frac{\lambda v}{1-v^2}\int_\tau^\infty e^{-\frac{(\tau'-\tau)}{v}}P(\tau')d\tau'.$$

The differential equation for $P(\tau)$ in the case when the boundary of the medium moves is

$$\frac{d^3P}{d\tau^3} - \frac{1}{v}\frac{d^2P}{d\tau^2} - \frac{dP}{d\tau} + \frac{1-\lambda}{v}P = 0. \tag{28}$$

This equation is easily obtained from (27) and also from (11) by changing the sign of the velocity v . Representing the function $P(\tau)$ in the form of (12), we find the characteristic equation from (28) . We find from the condition of boundedness of the solution as $\tau \to \infty$ that $c_1 = 0$, $c_3 = 0$, so that $k_1 < 0$ and $k_3 < 0$. Furthermore, substituting (12) into (27), we find the condition determining the

constant c_2:

(29) $$c_2 = (1 - k_2).$$

Hence, the expression for $P(\tau)$ in this case is

(30) $$P(\tau) = (1 - k_2)e^{-k_2\tau},$$

where

(31) $$k_2 = \frac{1}{3v}\left[2\sqrt{1+3v^2}\cos\left\{\frac{\pi}{3} - \frac{1}{3}\operatorname{arc\,cos}\frac{1}{2}\,\frac{2+9v^2(3\ \ -2)}{(1+3v^2)^{\frac{3}{2}}}\right\} - 1\right].$$

The approximate formulas for $v \ll 1$ are

(32) $$k_2 \approx \sqrt{1-\lambda} + \frac{(1-\lambda)v}{2},$$
$$P(\tau) \approx (1 - \sqrt{1-\lambda})e^{-\{\sqrt{1-\lambda} + \frac{(1-\lambda)v}{2}\}\tau};$$

for $v \gg 1$

(33) $$k_2 \approx 1 - \frac{\lambda}{2v},\quad P(\tau) \approx \frac{\lambda}{2v}e^{-\tau}.$$

Similarly, we also find an integral equation for $Z(\tau)$. Substituting (26) into (3), we obtain

(34) $$Z(\tau) = \frac{\lambda}{2(1+v)^2}e^{-\tau} + \frac{\lambda}{2(1+v)^2}\int_0^\tau e^{-(\tau-\tau')}P(\tau')d\tau' +$$
$$+ \frac{\lambda}{2(1-v)^2}\int_\tau^\infty e^{-(\tau'-\tau)}P(\tau')d\tau' - \frac{2\lambda v}{(1-v^2)^2}\int_\tau^\infty e^{-\frac{(\tau'-\tau)}{v}}P(\tau')d\tau' -$$
$$- \frac{\lambda}{1-v^2}\int_\tau^\infty(\tau'-\tau)e^{-\frac{(\tau'-\tau)}{v}}P(\tau')d\tau' + \frac{\lambda}{2(1+v)}\int_\tau^\tau e^{-(\tau-\tau')}Z(\tau')d\tau' +$$
$$+ \frac{\lambda}{2(1-v)}\int_\tau^\infty e^{-(\tau'-\tau)}Z(\tau')d\tau' - \frac{\lambda v}{1-v^2}\int_\tau^\infty e^{-\frac{(\tau'-\tau)}{v}}Z(\tau')d\tau'.$$

From this expression, we obtain the differential equation

(35) $$\frac{d^3Z}{d\tau^3} - \frac{1}{v}\cdot\frac{d^2Z}{d\tau^2} - \frac{dZ}{d\tau} + \frac{1-\lambda}{v}Z = \frac{1}{v}\left[P - \frac{d^2P}{d\tau^2}\right].$$

Substituting the solution for $Z(\tau)$ here in the form of (22), as before, we find that the coefficients B_i and A_i satisfy the conditions

$$\left.\begin{aligned} B_i &= \frac{1-k_i^2}{3k_i^2 v+2k_i-v}\, c_i \\ A_i &= \frac{1}{1-k_i}\, B_i \end{aligned}\right\} . \tag{36}$$

Taking into account that $c_1 = c_3 = 0$ in the case when the boundary moves outwards, we obtain the solution for $Z(\tau)$ as

$$Z(\tau) = \frac{1+k_2}{3k_2^2 v+2k_2-v}\,[1+(1-k_2)\tau]e^{-k_2\tau}, \tag{37}$$

which simplifies considerably for $v \gg 1$ and can be written as

$$Z(\tau) \approx \frac{1}{v}\left[1+\frac{\lambda}{2v}\tau\right] e^{-\tau}. \tag{38}$$

It is easy to see that each of the expressions and solutions obtained above transform for $v = 0$ into the known solutions found by V. V. Sobolev.

The question of applying the theory presented to practical problems related to the expansion of shockwaves in the terrestrial atmosphere will be examined separately.

Let us note that the equations and solutions presented here were partially published in [5].

In conclusion, the author is grateful to S. A. Kaplan for guidance during the research.

L'vov Univ. March 16, 1960

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